Topology and its Applications 21 (1985) 27-33
North-Holland

ALMOST ARCWISE CONNECTIVITY IN UNICOHERENT CONTINUA

E.E. GRACE
Department of Mathematics, Arizona State University, Tempe, AZ 85287, USA

C.L. HAGOPIAN*
Department of Mathematics and Statistics, California State University, Sacramento, CA 95819, USA

E.J. VOUGHT
Department of Mathematics, California State University, Chico, CA 95929, USA

Received 10 April 1984
Revised 8 October 1984

K.R. Kellum has proved that a continuum is an almost continuous image of the interval [0, 1] if and only if it is an almost Peano continuum. Hence, a continuum is an almost continuous image of [0, 1] if it has a dense arc component.

Our principal result is that any almost arcwise connected, semi-hereditarily unicoherent, metric continuum with only countably many arc components has a dense arc component. An example is given to show that this is not true for unicoherent continua in general. It is also shown that any semi-hereditarily unicoherent continuum with only countably many arc components has at most one dense arc component, and if it has a dense arc component, then every other arc component is nowhere dense. This generalizes results of Fugate and Mohler for λ-dendroids.

AMS (MOS) Subj. Class.: Primary 54F15, 54F20, 54F25, 54F55; Secondary 54G20

dense arc component	almost continuous function
almost arcwise connected continuum	almost Peano continuum
semi-hereditarily unicoherent continuum	unicoherent continuum

In [4], Kellum proved that a continuum is an almost continuous image of the interval [0, 1] if and only if it is an almost Peano continuum. Hence, a continuum is an almost continuous image of [0, 1] if it has a dense arc component.

Let M be an almost arcwise connected λ-dendroid with only countably many arc components. Then M has one dense arc component and every other arc component of M is nowhere dense [3, Theorem 3, p. 263]. Hence, M is an almost continuous image of [0, 1].

* Partially supported by NSF Grant MCS 8205282.

We generalize this theorem by replacing the assumption that M is a λ-dendroid with the assumption that M is a semi-hereditarily unicoherent continuum. An example is given to show that this result cannot be extended to unicoherent continua.

A *continuum* is a nondegenerate compact connected metric space. A continuum M is *almost arcwise connected* if for each pair of nonempty open subsets of M there is an arc that intersects both of them, and M is an *almost Peano continuum* if for each finite collection of nonempty open subsets of M, there is a Peano continuum (i.e. a locally connected continuum) in M that intersects each element of the collection. A continuum M is *semi-hereditarily unicoherent* [1] if $A \cap B$ is connected whenever A and B are subcontinua such that $\operatorname{Int}(A \backslash B) \neq \emptyset$ and $\operatorname{Int}(B \backslash A) \neq \emptyset$. Note that this definition is more general than that of weak hereditary unicoherence, due to Maćkowiak [7, p. 177] which requires $A \cap B$ to be connected whenever $\operatorname{Int}(A) \neq \emptyset$ and $\operatorname{Int}(B) \neq \emptyset$. A function $f: M \to N$ is *almost continuous* if each neighbourhood, in $M \times N$, of the graph of f contains the graph of a continuous function $g: M \to N$.

Lemma 1. *Suppose H and I are subcontinua of a semi-hereditarily unicoherent continuum M such that $H \cap I$ is not connected and* $\operatorname{Int}(H \backslash I) \neq \emptyset$. *If P is an arc component of M such that $P \cap (H \cap I) \neq \emptyset$ and P does not contain* $\operatorname{Int}(H \backslash I)$, *then P is first category in $M \backslash (H \cup I)$.*

Proof. Let $H \cap I = A \cup B$, a separation, and assume that P is second category in an open subset W of $M \backslash (H \cup I)$. Let x be an element of $[\operatorname{Int}(H \backslash I)] \backslash P$ and let $U_1, U_2, \ldots$ be a local basis for the topology at x in $\operatorname{Int}(H \backslash I)$. Let $P_A = \{t \in P \mid t \in A$ or there is an arc in $M \backslash B$ from t to $A\}$ and $P_B = \{t \in P \mid t \in B$ or there is an arc in $M \backslash A$ from t to $B\}$. Since $P \cap (H \cap I) \neq \emptyset$, clearly $P = P_A \cup P_B$, so either P_A or P_B is second category in W. Assume without loss of generality that P_A is second category in W. For each natural number i, let N_i be the $1/i$-neighborhood of B and let $P_i = \{t \in P_A \mid t \in A$ or there is an arc in $M \backslash (U_i \cup N_i)$ from t to a point of $A\}$. Then $P_A = \bigcup_{i=1}^{\infty} P_i$, hence, for some natural number j, $\operatorname{cl} P_j$ contains an open set in W. Let $K = I \cup \operatorname{cl} P_j$, which is a continuum. Then K contains an open set in W but does not contain $\operatorname{Int}(H \backslash I)$, H contains $\operatorname{Int}(H \backslash I)$ but does not intersect W and

$$\begin{aligned} H \cap K &= H \cap (I \cup \operatorname{cl} P_j) = (H \cap I) \cup (H \cap \operatorname{cl} P_j) \\ &= (A \cup B) \cup (H \cap \operatorname{cl} P_j) = B \cup (A \cup (H \cap \operatorname{cl} P_j)). \end{aligned}$$

But B and $A \cup (H \cap \operatorname{cl} P_j)$ are disjoint closed sets and this contradicts the semi-hereditary unicoherence of M. $\square$

Theorem 2. *If M is a semi-hereditarily unicoherent continuum and P is an arc component of M that is densely second category in an open subset U of M, then any other arc component is nowhere dense in U.*

Proof. Assume another arc component Q is dense in an open subset U' of U. Let $p \in P$ and $q \in Q$. Let I be an irreducible subcontinuum of M from p to q. Without loss of generality, it can be assumed that $I \cap U' = \emptyset$. To see this, let $r \in Q \cap U'$ and let $B_1, B_2, \ldots$ be a local basis at r in U'. For $i = 1, 2, \ldots$, let P_i be the p-component of $M \setminus B_i$. Since $P \subseteq \bigcup_{i=1}^{\infty} P_i$ and P is second category in U', there is a natural number j such that P_j contains an open subset V of U'. Since Q is dense in V, there is an arc A from q to a point of V and since P is dense in B_j, $B_j \not\subseteq A$. Let I be a subcontinuum of $P_j \cup A$ that is irreducible from p to q and rename U' to be the open set $B_j \setminus A$. Then $I \cap U' = \emptyset$ as claimed.

Since I is irreducible but is not an arc, I is not locally connected at some two points r and s not including p or q.

Case 1. One of those points, r, belongs to an arc component different from P.

Choose a point q' in $Q \cap U'$, and let $B_1, B_2, \ldots$ be a local basis in U' at q', let $D_1, D_2, \ldots$ be a local basis at r, and, for $i = 1, 2, \ldots$, let P_i be the p-component of $M \setminus (B_i \cup D_i)$. Since P is second category in U' and $P \subseteq \bigcup_{i=1}^{\infty} P_i$, there is an integer j such that P_j contains an open set V in U'. Since Q is dense in V, there is an arc A from q to a point of V. Since P is dense in B_j, A does not contain B_j. Let $B_j' = B_j \setminus A$. Also, A does not contain $D_j \cap I$, since I is not locally connected at r. Let $D_j' = D_j \setminus A$. The continuum $H = P_j \cup A$ contains p and q but $H \cap D_j' = \emptyset$ and so, since I is irreducible from p to q, $H \cap I$ is not connected. Then by Lemma 1, P is first category in B_j'. But $B_j' \subseteq U'$ and P is densely second category in U', a contradiction.

Case 2. Both points, r and s, belong to P.

Let $P_r = \{t \in P \mid t \text{ is in the } r \text{ arc component of } M \setminus \{s\}\}$ and $P_s = \{t \in P \mid t \text{ is in the } s \text{ arc component of } M \setminus \{r\}\}$. Then $P = P_r \cup P_s$. Hence, without loss of generality, $P_s \cap U'$ is second category. Choose a point q' in $Q \cap U'$, and let $B_1, B_2, \ldots$ be a local basis at q' in U', let $D_1, D_2, \ldots$ be a local basis at r, and, for $i = 1, 2, \ldots$, let P_i be the s-component of $M \setminus (B_i \cup D_i)$. Since P_s is second category in U' and $P_s \subseteq \bigcup_{i=1}^{\infty} P_i$, there is an integer j such that P_j contains an open subset V of U'. Since Q is dense in V there is an arc A from q to a point of V. Also, let A' be an arc from p to s. Since I is not locally connected at r, $A \cup A'$ does not contain $D_j \cap I$. Let $D_j' = D_j \setminus (A \cup A')$. Since P is dense in B_j, A does not contain B_j. Similarly, since Q is dense in B_j, A' does not contain $B_j \setminus A$. Let $B_j' = B_j \setminus (A \cup A')$. The continuum $H = P_j \cup A \cup A'$ contains p and q, but $H \cap D_j' = \emptyset$. Since I is irreducible from p to q, $H \cap I$ is not connected. Then, by Lemma 1, P is first category in B_j'. But $B_j' \subseteq U'$ and P is densely second category in U', a contradiction. □

Corollary 3. *If M is a semi-hereditarily unicoherent continuum with a dense arc component, then every other arc component is first category in M.*

The following corollary generalizes results of Fugate and Mohler for λ-dendroids [2, Corollaries 1.9 and 1.10, pp. 396, 397]. Examples are given in [5] and [6] which show that the condition that M have countably many arc components cannot be omitted (even for λ-dendroids).

Corollary 4. *Suppose M is a semi-hereditarily unicoherent continuum that has only countably many arc components. Then M has at most one dense arc component and if M has a dense arc component, then every other arc component is nowhere dense.*

Proof. Since M has a countable number of arc components, there exists an arc component R that is densely second category in an open subset U of M. If P is a dense arc component of M, then by Theorem 2, $P = R$. Hence, M has at most one dense arc component.

Now suppose P is a dense arc component of M and Q is an arc component that is dense in an open set U. Since M has a countable number of arc components, there exists an arc component R that is densely second category in an open subset U' of U. By Theorem 2, $R = Q$ and $R = P$, hence $P = Q$. Therefore, any arc component of M other than P is nowhere dense, and Corollary 4 is proved. □

Theorem 5. *Suppose M is an almost arcwise connected, semi-hereditarily unicoherent continuum. Then at most one arc component of M is second category.*

Proof. Suppose M has two second category arc components, P and Q. Then P and Q are densely second category in some open sets U and V, respectively, which by Theorem 2, can be chosen so that cl $P \cap V = \emptyset$ and cl $Q \cap U = \emptyset$. Let I be a subcontinuum of M that is irreducible between a point p of P and a point q of Q. It can be assumed without loss of generality that $I \cap (U \cup V) = \emptyset$. To see this, let A_1 be an arc from a point r of U to a point s of V. Let $R_1, R_2, \ldots$ and $S_1, S_2, \ldots$ be point bases at r and s, respectively, in U and V, respectively. For each natural number i, let P_i and Q_i be the p-component and q-component, respectively, of $M \backslash (R_i \cup S_i)$. Then $\bigcup_{i=1}^{\infty} P_i \supseteq P$ and $\bigcup_{i=1}^{\infty} Q_i \supseteq Q$, so, for some natural number j, P_j and Q_j contain open subsets of U and V, respectively. Let A_2 be an arc from a point of P_j to a point of Q_j, and let I be a subcontinuum of $P_j \cup A_2 \cup Q_j$ that is irreducible from p to q. Also, reassign the names 'U' and 'V' to $R_j \backslash A_2$ and $S_j \backslash A_2$, respectively. Then U, V and I have the cited properties.

Since I is irreducible but is not an arc, I is not locally connected. Let r be a point of $I \backslash \{p, q\}$ at which I is not locally connected. Then, without loss of generality, $r \notin Q$. Let $R_1, R_2, \ldots,$ be a local basis at r. For some natural number j, q', the closure of the q arc component of $M \backslash R_j$, contains an open subset of V. Also $q' \subseteq M \backslash U$. Let M' be the continuum resulting from 'shrinking' q' to a point. Then M' is semi-hereditarily unicoherent, since q' is a continuum. Since M is almost arcwise connected, the q' arc component of M' is dense in M' and, hence, is dense in U. But P', the p arc component of M', is densely second category in U. Hence, by Theorem 2, $q' \in P'$. Let A' be an arc from p to q' in M'. Since I is not locally connected at r, $R_j \cap I$ is not contained in any arc. Hence, $H = q' \cup (A' \backslash \{q'\})$ is a continuum in M that does not contain I. Therefore, since I is irreducible from p to q and $\{p, q\} \subseteq H$, $H \cap I$ is not connected. It follows, by Lemma 1, that P is first category in $M \backslash (H \cup I)$ and hence, in U, a contradiction. □

In [3, Theorem 3, p. 263] an incorrect proof was given in the setting where M is a λ-dendroid. The following theorem offers a correct proof in the more general setting where M is semi-hereditarily unicoherent. [3, Example 3, p. 262] shows that the condition that M have only countably many arc components cannot be omitted even for almost Peano λ-dendroids.

Theorem 6. *Suppose M is a semi-hereditarily unicoherent continuum that is almost arcwise connected and has only countably many arc components. Then M has a dense arc component and every other arc component is nowhere dense in M.*

Proof. Let U and V be any two open subsets of M. Since M is second category in U and in V and is the union of countably many arc components, some arc component, P, is second category in U and some arc component, Q, is second category in V. By Theorem 5, $P=Q$. But U and V are any two open subsets of M. Hence, M contains a dense arc component that is, in fact, second category in every open subset of M. By Corollary 4, every other arc component is nowhere dense in M. □

The example that follows shows that Theorem 6 cannot be extended to unicoherent continua, even to hereditarily decomposable ones.

Example 7. There exists a unicoherent, hereditarily decomposable, almost Peano continuum Z that has only countably many arc components and does not have a dense arc component.

To define Z, let C be the Cantor ternary set in the unit interval $[0,1]$. For $i=1,2,\ldots,$ let

$$E_i=\left\{\frac{0}{3^i},\frac{1}{3^i},\ldots,\frac{3^i}{3^i}\right\}\cap C,$$

and let

$$P_i=\left\{(x,y,0)\in E^3 \mid x\in E_i \text{ and } \frac{1}{2i+1}\leqslant y\leqslant\frac{1}{2i}\right\}.$$

In E^3 uniformly modify $C\times[0,1]\times\{0\}$ so that each component of $\bigcup_{i=1}^{\infty}P_i$ is the limit bar of a $\sin 1/x$ curve as indicated in Fig. 1. For x in $C\backslash[\bigcup_{i=1}^{\infty}E_i]$ the arc $\{x\}\times[0,1]\times\{0\}$ is modified so that it remains an arc, and so that the collection of components of the resulting space W is a continuous collection. Hence, W is compact, and the components of W that miss $\bigcup_{i=1}^{\infty}P_i$ are arcs whose union is dense in W.

For $i=1,2,\ldots,$ let

$$Q_i=\left\{(x,y,z)\in W \mid y\leqslant\frac{1}{2i} \text{ and } x\in E_i\right\}.$$

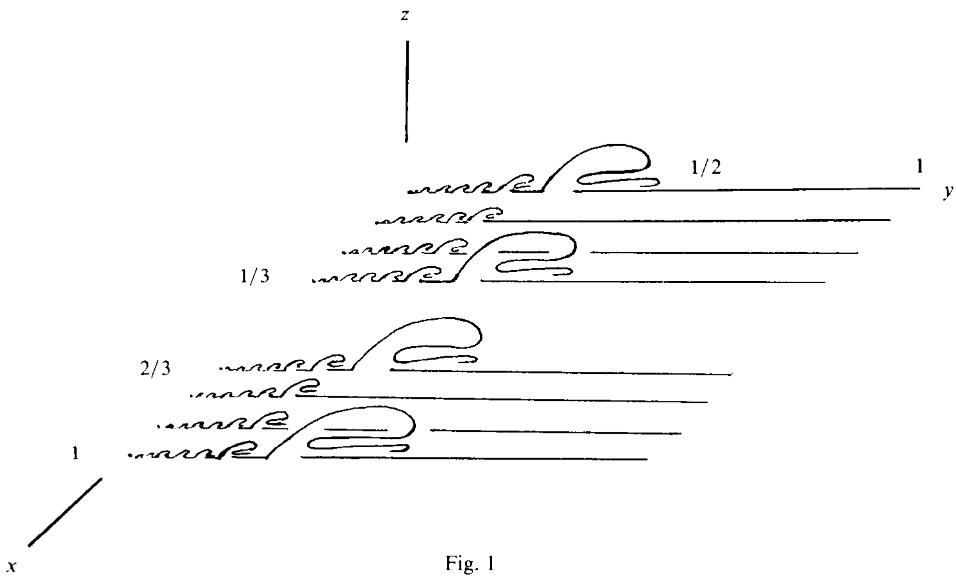

Fig. 1

Let d be the function from W into W defined by

$$d(x, y, z) = \begin{cases} (x, y, z) & \text{if } (x, y, z) \in W \backslash [\bigcup_{i=1}^{\infty} Q_i], \\ (0, y, z) & \text{if } (x, y, z) \in \bigcup_{i=1}^{\infty} Q_i \text{ or } y = 0. \end{cases}$$

Let X be $d[W]$ with the quotient topology. Note that X is a continuum that has only countably many arc components and only one of them is dense in X.

Let h be a homeomorphism on X such that $X \cap h[X] = \emptyset$. Let $A_1, A_2, \ldots$ be a convergent sequence of disjoint arcs with the following properties. For each i, the end points of A_i are $(0, 1/2i, 0)$ and $h(0, 1/2i, 0)$ and no other point of A_i belongs to $X \cup h[X]$. The limit S of $A_1, A_2, \ldots$ is a $\sin 1/x$ curve with limit bar B such that $B \cap X = \emptyset$, $S \cap X = \{(0, 0, 0)\}$, and $B \cap h[X] = S \cap h[X] = \{h(0, 0, 0)\}$. Note that $(0, 0, 0)$ and $h(0, 0, 0)$ are in different arc components of S.

Let Y be $X \cup h[X] \cup S \cup [\bigcup_{i=1}^{\infty} A_i]$. Note that Y has only countably many arc components and does not have a dense arc component. Also, if D_1 and D_2 are open subsets of X and $h[X]$, respectively, then some arc in Y intersects D_1 and D_2. However, Y is not unicoherent, and is not an almost Peano continuum, because $S \cup [\bigcup_{i=1}^{\infty} A_i]$ has nonvoid interior.

Let μ be the Hausdorff metric on 2^Y (the space of closed subsets of Y). Let $G_1, G_2, \ldots$ be a sequence of disjoint nonempty open sets in $W \backslash \bigcup_{i=1}^{\infty} Q_i$ such that $\mu(\{(0, 0, 0)\}, \text{cl } G_i) \leq 1/i$ for each i. For each i, let H_i be the $\sin 1/x$ curve in X that is irreducible between A_i and A_{i+1}, and let J_i be $A_i \cup H_i \cup h[H_i]$. Let $K = \text{cl}[\bigcup_{i=1}^{\infty} J_i]$.

Leaving $Y \backslash G_1$ fixed, stretch G_1 around K so that $\mu(K, \text{cl } G_1) \leq 1$. Continue this process. For each i, without moving $Y \backslash G_i$, stretch G_i so that $\mu(K, \text{cl } G_i) \leq 1/i$. The result of this process is the desired continuum Z.

Since Z is an almost Peano continuum, Z is an almost continuous image of $[0, 1]$.

Question 8. Is every unicoherent, almost arcwise connected continuum that has only countably many arc components an almost Peano continuum?

References

[1] W.D. Collins and E.J. Vought, Continuum chainability and monotone decompositions in certain classes of unicoherent continua, Houston J. Math. 10 (1984) 457-465.
[2] J.B. Fugate and L. Mohler, The fixed point property for tree-like continua with finitely many arc components, Pacific J. Math. 57 (1975) 393-402.
[3] C.L. Hagopian, Almost arcwise connected continua without dense arc components, Topology Appl. 12 (1981) 257-265.
[4] K.R. Kellum, Almost continuous images of Peano continua, Topology Appl. 11 (1980) 293-296.
[5] J. Krasinkiewicz and P. Minc, Approximations of continua from within, Bull. Acad. Polon. Sci. 25 (1977) 283-289.
[6] L. Mohler, Arc components and density in λ-dendroids, Houston J. Math. 8 (1982) 109-117.
[7] T. Maćkowiak, On some characterizations of dendroids and weakly monotone mappings, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 24 (1976) 177-182.